Narrowband Noise

Channel Model *Consider a linear, time-invariant channel whose transfer function,* $H(f)$*, is identically zero outside a band of frequencies of total width* $2W$ *around a centre frequency* f_c*:*

$$H(f) \equiv 0 \qquad \textit{if } |f - f_c| > W.$$

As usual, we assume that $f_c > 2W$*. The channel impulse response,* $h(t) \overset{\mathcal{F}}{\rightleftharpoons} H(f)$*, is hence a narrowband function of bandwidth* $2W$ *and centre frequency* f_c*. We will assume that the channel is noisy; namely, that any signal transmitted through the channel will be corrupted by a random interference, or noise.*

Noise Model *We will consider channel noise to be additive. In particular, if* $s(t)$ *denotes a transmitted signal, then the received signal will be of the form*

$$\begin{aligned} r(t) = \big(s(t) + n_w(t)\big) \star h(t) &= s(t) \star h(t) + n_w(t) \star h(t) \\ &= \int_{-\infty}^{\infty} s(t-\tau)h(\tau)\,d\tau + \int_{-\infty}^{\infty} n_w(t-\tau)h(\tau)\,d\tau, \end{aligned}$$

where $n_w(t)$ *is an additive random noise process generated in the channel. We will assume that* $n_w(t)$ *is a zero mean, stationary, white Gaussian process with power spectral density* $S_{n_w}(f) = N_0/2$*.* In addition we assume that the channel noise process* $n_w(t)$ *is independent of the signal* $s(t)$*.†*

The additive white noise $n_w(t)$ *generated in the bandlimited communication channel results in a noise process,* $N(t) = n_w(t) \star h(t)$*, which additively contaminates the received signal. Clearly,* $N(t)$ *is wide sense stationary, zero mean process with power spectral density*

$$S_N(f) = \tfrac{N_0}{2}\,\big|H(f)\big|^2.$$

As the channel is narrowband, $S_N(f)$ *is also narrowband with bandwidth* $2W$ *and centre frequency* f_c*. We hence refer to* $N(t)$ *as a narrowband process.*

**Whiteness is a reasonable assumption for certain natural noise sources—a classical instance is the white noise you see on a TV screen; this is not, however, a good assumption for man made clutter, intelligent jammers, etc. In cases where the noise is not white, common terminology refers to the noise as coloured.*

The zero mean assumption is not particularly confining for additive noise processes. If the process has a nonzero mean μ*, then the receiver sees an additive noise component whose mean is* $\mu H(0)$*. Clearly, this additive mean noise component can be subtracted away resulting in a zero mean noise process. The Gaussian assumption is also standard, though harder to justify. It does make the analysis of noise substantially easier, and one can loosely support it by appealing vaguely to the Central Limit Theorem.*

†*The additive, signal-independent noise assumption is common, and fits channels like the atmosphere rather well. You should be cognisant, however, that there are situations in practice where one or both assumptions are invalid. In particular, there are instances where multiplicative (and other nonadditive models) better describe the noisy interference, and situations where the noise may be signal-dependent. An example where a nonadditive, signal-dependent noise model is appropriate is in characterising noise in certain classes of photographically generated images. Another example of nonadditive noise arises in quantisation e ects in digital modulation schemes such as pulse code modulation.*

Representation *Just as in the representation of bandlimited signals, we can use an in-phase and quadrature representation or an envelope and phase representation to characterise the bandpass process* $N(t)$ *in terms of lowpass processes,*

$$N(t) = \begin{cases} N_I(t)\,cos(2\pi f_c t) - N_Q(t)\,sin(2\pi f_c t) & \textit{(in-phase and quadrature representation),} \\ \mathfrak{R}(t)\,cos\big[2\pi f_c t + \Psi(t)\big] & \textit{(envelope and phase representation),} \end{cases}$$

where, in analogy with the representation for deterministic narrowband functions, $N_I(t)$ *and* $N_Q(t)$ *represent the in-phase and quadrature components, respectively, of the noise process, and* $\mathfrak{R}(t)$ *and* $\Psi(t)$ *represent the corresponding envelope and phase processes. As usual, these lowpass processes are obtained from the bandpass process* $N(t)$ *by frequency shifting and lowpass filtering. Write*

$$h_L(t) = 2W\,sinc(2Wt) \overset{\mathcal{F}}{\rightleftharpoons} H_L(f) = rect\big(\tfrac{f}{2W}\big)$$

for the impulse response and transfer function of an ideal lowpass filter with bandwidth W*. Here, in standard notation, we identify the Fourier pairs*

$$sinc(t) = \begin{cases} 1 & if\ t = 0, \\ \frac{sin(\pi t)}{\pi t} & if\ t \neq 0, \end{cases} \qquad \overset{\mathcal{F}}{\rightleftharpoons} \qquad rect(f) = \begin{cases} 1 & if\ |f| < 1/2, \\ 0 & if\ |f| \geq 1/2. \end{cases}$$

It is easy to verify that

$$sinc(t) = \int_{-\infty}^{\infty} rect(f)e^{j2\pi ft}\,df,$$

so that

$$rect(f) = \int_{-\infty}^{\infty} sinc(t)e^{-j2\pi ft}\,dt,$$

as well, by the uniqueness of the Fourier transform. The in-phase and quadrature components and the envelope and phase processes of the noise process $N(t)$ *are then given by*

$$\begin{aligned} N_I(t) &= +2\big[N(t)\,cos(2\pi f_c t)\big] \star h_L(t) && \textit{(in-phase component),} \\ N_Q(t) &= -2\big[N(t)\,sin(2\pi f_c t)\big] \star h_L(t) && \textit{(quadrature component),} \\ \mathfrak{R}(t) &= \sqrt{N_I^2(t) + N_Q^2(t)} && \textit{(envelope process),} \\ \Psi(t) &= arctan\left[\frac{N_Q(t)}{N_I(t)}\right] && \textit{(phase process).} \end{aligned}$$

What can we say about the joint statistics of the processes $N_I(t)$ *and* $N_Q(t)$*, or, equivalently,* $\mathfrak{R}(t)$ *and* $\Psi(t)$*?*

Properties *To begin with, observe that* $N(t)$ *is a zero mean, stationary, Gaussian process. (Why?)*

Property 1 *The in-phase and quadrature components,* $N_I(t)$ *and* $N_Q(t)$*, are zero mean, jointly Gaussian random processes.* Why?

Property 2 *The processes* $N_I(t)$ *and* $N_Q(t)$ *are jointly stationary.*

Proof: *Let's begin by showing that the in-phase and quadrature noise processes are individually stationary. Consider the process* $N_I(t)$ *first. It suffices to show* $N_I(t)$ *is wide sense stationary. (Why?) Even though we are passing the noise process through a linear (low-pass) filter, we cannot directly claim that (wide sense) stationarity is preserved because of the multiplying factor* $\cos(2\pi f_c t)$*. (In fact,* $N(t)\cos(2\pi f_c t)$ *is not wide sense stationary. Verify.) We have*

$$\begin{aligned} &E\big[N_I(t+\tau)N_I(t)\big] \\ &= 4\iint_{-\infty}^{\infty} E\big[N(t+\tau-x)N(t-y)\big]\cos\big(2\pi f_c(t+\tau-x)\big)\cos\big(2\pi f_c(t-y)\big)h_L(x)h_L(y)\,dx\,dy \\ &= 2\iint_{-\infty}^{\infty} R_N(\tau-x+y)\big[\cos\big(2\pi f_c(\tau-x+y)\big)+\cos\big(2\pi f_c(2t+\tau-x-y)\big)\big]h_L(x)h_L(y)\,dx\,dy. \end{aligned}$$

With the coordinate transformations $x-y=u$, $x+y=v$*, we obtain*

$$\begin{aligned} &E\big[N_I(t+\tau)N_I(t)\big] \\ &= \overbrace{\iint_{-\infty}^{\infty} R_N(\tau-u)\cos\big(2\pi f_c(\tau-u)\big)h_L\big(\tfrac{v+u}{2}\big)h_L\big(\tfrac{v-u}{2}\big)\,dv\,du}^{A} \\ &\quad + \underbrace{\iint_{-\infty}^{\infty} R_N(\tau-u)\cos\big(2\pi f_c(2t+\tau-v)\big)h_L\big(\tfrac{v+u}{2}\big)h_L\big(\tfrac{v-u}{2}\big)\,dv\,du}_{B}. \qquad (*) \end{aligned}$$

The first integral yields a result depending purely on the shift τ*. For wide sense stationarity it hence suffices to show that the second integral evaluates to a quantity independent of* t*. Integrating along the* v*-variable first gives*

$$\begin{aligned} &\int_{-\infty}^{\infty} h_L\big(\tfrac{v+u}{2}\big)h_L\big(\tfrac{v-u}{2}\big)\cos\big(2\pi f_c(v-2t-\tau)\big)\,dv \\ &= 2\int_{-\infty}^{\infty} h_L\big(\omega+\tfrac{u}{2}\big)h_L\big(\omega-\tfrac{u}{2}\big)\cos\big(2\pi f_c(2\omega-2t-\tau)\big)\,d\omega \\ &= 2\,\mathrm{Re}\Big\{e^{j2\pi f_c(2t+\tau)}\int_{-\infty}^{\infty} h_L\big(\omega+\tfrac{u}{2}\big)h_L\big(\omega-\tfrac{u}{2}\big)e^{-j2\pi(2f_c)\omega}\,d\omega\Big\} \\ &= 2\,\mathrm{Re}\Big\{e^{j2\pi f_c(2t+\tau)}\int_{-\infty}^{\infty} H_L(f)e^{j\pi fu}\,H_L(2f_c-f)\,e^{-j\pi(2f_c-f)u}\,df\Big\} \\ &= 2\,\mathrm{Re}\Big\{e^{j2\pi f_c(2t+\tau-u)}\int_{-\infty}^{\infty} \mathrm{rect}\big(\tfrac{f}{2W}\big)\,\mathrm{rect}\big(\tfrac{2f_c-f}{2W}\big)e^{j2\pi fu}\,df\Big\}, \end{aligned}$$

where the penultimate step follows by noting that the ω*-integral is simply the Fourier transform of shifted products of* h_L *evaluated at a "frequency"* $2f_c$*. For* $f_c > W$ *there is no overlap of the two rect functions, and the integral is identically zero for each value of* u*. (The particular choices of* t *and* τ *also do not affect this.) Hence,* $B \equiv 0$*.*

It follows that $E\big[N_I(t+\tau)N_I(t)\big] = A = R_{N_I}(\tau)$ *so that* $N_I(t)$ *is wide sense stationary, hence stationary vide* Property 1*. A similar argument serves for* $N_Q(t)$*.*

To complete the proof of the assertion we need to show that $N_I(t)$ *and* $N_Q(t)$ *are jointly stationary. Again, it suffices to show that* $N_I(t)$ *and* $N_Q(t)$ *are jointly wide* *sense stationary. (Why?) In particular, it suffices to show that* $N_I(t)$ *and* $N_Q(t+\tau)$ *are uncorrelated random variables for any choice of* t *and* τ*. (Why?) Proceeding as before, in analogy with* $(*)$*, it is not difficult to obtain the expression*

$$\begin{aligned}
&\mathbf{E}\big[N_Q(t+\tau)N_I(t)\big] \\
&\quad = \overbrace{-\iint_{-\infty}^{\infty} R_N(-u+\tau)\, sin\big(2\pi f_c(\tau-u)\big) h_L\big(\tfrac{v+u}{2}\big) h_L\big(\tfrac{v-u}{2}\big)\, dv\, du}^{C} \\
&\qquad + \underbrace{\iint_{-\infty}^{\infty} R_N(-u+\tau)\, sin\big(2\pi f_c(2t+\tau-v)\big) h_L\big(\tfrac{v+u}{2}\big) h_L\big(\tfrac{v-u}{2}\big)\, dv\, du}_{D}. \qquad (**)
\end{aligned}$$

By reasoning similar to that for the B-integral in $(*)$ *we can demonstrate that* $D \equiv 0$*. Observe that* C *depends only on* τ *and not on* t*. Consequently, the cross-correlation* $\mathbf{E}\big[N_Q(t+\tau)N_I(t)\big] = C = R_{N_Q N_I}(\tau)$ *depends only on* τ *and not on* t *so that we have shown that the processes* $N_I(t)$ *and* $N_Q(t)$ *are jointly wide sense stationary.* ▮

Property *3 The processes* $N_I(t)$ *and* $N_Q(t)$ *are lowpass (with spectral bandwidth W), and have the same power spectral density*

$$S_{N_I}(f) = S_{N_Q}(f) = \big[S_N(f-f_c) + S_N(f+f_c)\big]\, rect\big(\tfrac{f}{2W}\big).$$

Proof*: Let's evaluate* A *in* $(*)$ *explicitly. We have*

$$R_{N_I}(\tau) = \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} h_L\big(\tfrac{v+u}{2}\big) h_L\big(\tfrac{v-u}{2}\big)\, dv\right)}_{P(u)} R_N(\tau-u)\, cos\big(2\pi f_c(\tau-u)\big)\, du$$

Evaluating the inner integral first, we have

$$\begin{aligned}
P(u) &= 2\int_{-\infty}^{\infty} h_L(z)h_L(z-u)\, dz && \textit{(substitute } z=(u+v)/2) \\
&= 2\int_{-\infty}^{\infty} h_L(z)h(u-z)\, dz && \textit{(as } h_L(z) = 2W\, sinc(2Wz) \textit{ is even)} \\
&= 2\int_{-\infty}^{\infty} H_L^2(f)e^{j2\pi fu}\, df && \textit{(Fourier convolution relation)} \\
&= 2\int_{-\infty}^{\infty} H_L(f)e^{j2\pi fu}\, df && \textit{(as } H_L^2(f) = H_L(f) = rect(f/2W)) \\
&= 2h_L(u) && \textit{(inverse Fourier transform).} \qquad (***)
\end{aligned}$$

It follows that

$$R_{N_I}(\tau) = 2\int_{-\infty}^{\infty} h_L(u)R_N(\tau-u)\, cos\big(2\pi f_c(\tau-u)\big)\, du.$$

We recognise the right-hand side as the convolution of $h_L(\tau)$ *with* $R_N(\tau)\, cos(2\pi f_c\tau)$*. It follows that the power spectral density* $S_{N_I}(f)$ *is the product of* $H_L(f)$ *with the Fourier transform* $\mathcal{F}\{R_N(\tau)\, cos(2\pi f_c\tau)\}$*. Hence,*

$$S_{N_I}(f) = 2H_L(f)\left[S_N(f) \star \left\{\tfrac{\delta(f-f_c)+\delta(f+f_c)}{2}\right\}\right] = rect\big(\tfrac{f}{2W}\big)\big[S_N(f-f_c) + S_N(f+f_c)\big].$$

An identical derivation yields the same result for $S_{N_Q}(f)$. ∎

Property 4 *The noise processes* $N(t)$, $N_I(t)$, *and* $N_Q(t)$ *all have the same power:* Why?

$$\mathit{Var}\, N_I(t) = \mathit{Var}\, N_Q(t) = \mathit{Var}\, N(t) \triangleq \sigma_N^2.$$

Let us introduce some terminology: say that $S_N(f)$ *is locally symmetric about* $\pm f_c$ *if*

$$S_N(f_c + f) = S_N(f_c - f) \quad \textit{and} \quad S_N(-f_c + f) = S_N(-f_c - f) \qquad \textit{for } 0 \le f \le W.$$

Property 5 *If* $S_N(f)$ *is locally symmetric about* $\pm f_c$, *then* $N_I(t)$ *and* $N_Q(t)$ *are independent processes.*

Proof: *Let's evaluate* C *in* $(**)$ *explicitly (recall that* $D = 0$*). Evaluating the* ν*-integral first using* $(***)$, *we obtain the following expression for the cross-correlation function:*

$$R_{N_Q N_I}(\tau) = -2\int_{-\infty}^{\infty} h_L(u) R_N(\tau - u)\, \mathit{sin}\big(2\pi f_c(\tau - u)\big)\, du.^{\ddagger}$$

Looking at the Fourier transform of $R_{N_Q N_I}(\tau)$, *i.e., the cross-spectral density* $S_{N_Q N_I}(f)$, *we obtain*

$$S_{N_Q N_I}(f) = \mathcal{F}\big\{R_{N_Q N_I}(\tau)\big\} = j\, \mathit{rect}\big(\tfrac{f}{2W}\big)\big[S_N(f - f_c) - S_N(f + f_c)\big].^{\S}$$

Now, power spectral densities are symmetric, so that $S_N(f_c - f) = S_N(f - f_c)$*; the local symmetry of* $S_N(f)$ *hence yields* $S_N(f_c + f) = S_N(f - f_c)$. *It follows that* $S_{N_Q N_I}(f) \equiv 0$, *whence* $R_{N_Q N_I}(\tau) = \mathcal{F}^{-1}\big\{S_{N_Q N_I}(f)\big\} = 0$. *As* $N_I(t)$ *and* $N_Q(t)$ *are zero mean, it follows that*

$$0 = \mathbf{E}\big[N_Q(t + \tau) N_I(t)\big] = \mathbf{E}\big[N_Q(t + \tau)\big]\, \mathbf{E}\big[N_I(t)\big] = 0.$$

As t *and* τ *are arbitrary,* $N_Q(t)$ *and* $N_I(t)$ *are uncorrelated processes.* ∎

The next property concerns the distributions of the envelope and phase noise processes, $\mathfrak{R}(t)$ *and* $\Psi(t)$, *respectively.*

Property 6 *If* $S_N(f)$ *is locally symmetric about* $\pm f_c$, *then, for every* t, *the random variables* $\mathfrak{R}(t)$ *and* $\Psi(t)$ *are independent. The marginal distributions are moreover independent of* t *with* $\mathfrak{R}(t)$ *possessing a Rayleigh distribution,*

$$p_{\mathfrak{R}}(r) = \begin{cases} \frac{r}{\sigma_N^2}\, e^{-r^2/2\sigma_N^2}, & \textit{if } r \ge 0, \\ 0, & \textit{if } r < 0, \end{cases}$$

while $\Psi(t)$ *is uniform on* $[0, 2\pi)$,

$$p_{\Psi}(\psi) = \begin{cases} \frac{1}{2\pi}, & \textit{if } 0 \le \psi < 2\pi, \\ 0, & \textit{otherwise}. \end{cases}$$

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‡ *Note that the cross-correlation exhibits skew-symmetry,* $R_{N_Q N_I}(\tau) = -R_{N_I N_Q}(\tau)$, *i.e., the cross-correlation is an odd function of* τ, *as a consequence of the fact that sin is an odd function while* R_N *and* h_L *are even functions.*

§ *Note that the cross-spectral density has no power interpretation; hence, is not required to be real, nonnegative.*

Remark*: Observe that we write simply* $p_{\mathfrak{R}}(r)$ *and* $p_\Psi(\psi)$ *instead of* $p_{\mathfrak{R}(t)}(r)$ *and* $p_{\Psi(t)}(\psi)$*, respectively, as the marginal densities do not depend on* t.

Proof*: Under the given conditions,* $N_I(t)$ *and* $N_Q(t)$ *are jointly Gaussian, independent, and have the same variance* σ_N^2*; consequently, their joint pdf is*

$$p_{N_I,N_Q}(x,y) = \frac{1}{2\pi\sigma_N^2} e^{-(x^2+y^2)/2\sigma_N^2}.$$

Consider the joint distribution function

$$\begin{aligned} F_{\mathfrak{R},\Psi}(r,\psi) &= \mathbf{P}\{0 \le \mathfrak{R}(t) \le r,\ 0 \le \Psi(t) \le \psi\} \\ &= \mathbf{P}\left\{0 \le \sqrt{N_I^2(t)+N_Q^2(t)} \le r,\ 0 \le \arctan\left(\frac{N_Q(t)}{N_I(t)}\right) \le \psi\right\} \\ &= \iint \frac{1}{2\pi\sigma_N^2} e^{-(x^2+y^2)/2\sigma_N^2}\,dx\,dy, \end{aligned}$$

where the double integral ranges over the sector

$$\left\{ (x,y) : 0 \le \sqrt{x^2+y^2} \le r,\ 0 \le \arctan\left(\tfrac{y}{x}\right) \le \psi \right\}$$

in the circle of radius r *in the two-dimensional Cartesian plane. With the usual Cartesian to polar transformation* $\rho = \sqrt{x^2+y^2}$, $\phi = \arctan(y/x)$*, we have*

$$F_{\mathfrak{R},\Psi}(r,\psi) = \int_{\rho=0}^{r}\int_{\phi=0}^{\psi} \frac{1}{2\pi\sigma_N^2} e^{-\rho^2/2\sigma_N^2}\rho\,d\phi\,d\rho = \frac{\psi}{2\pi}\left[1 - e^{-r^2/2\sigma_N^2}\right]$$

for every choice of $0 \le r < \infty$ *and* $0 \le \psi < 2\pi$ *. (While the above holds, strictly speaking, only for the first quadrant, by symmetry we can extend it for all* ψ *in* $[0,2\pi)$*.)*

The marginal distribution functions of $\mathfrak{R}(t)$ *and* $\Psi(t)$ *can be obtained by setting extreme values,* $\psi = 2\pi$*, and* $r = +\infty$*, respectively, in* $F_{\mathfrak{R},\Psi}(r,\psi)$*:*

$$\begin{aligned} F_{\mathfrak{R}}(r) &= F_{\mathfrak{R},\Psi}(r,\psi)\big|_{\psi=2\pi} = 1 - e^{-r^2/2\sigma_N^2} && (0 \le r < \infty), \\ F_\Psi(\psi) &= F_{\mathfrak{R},\Psi}(r,\psi)\big|_{r=+\infty} = \frac{\psi}{2\pi} && (0 \le \psi < 2\pi\ ). \end{aligned}$$

We hence have $F_{\mathfrak{R},\Psi}(r,\psi) = F_{\mathfrak{R}}(r)F_\Psi(\psi)$*, so that* $\mathfrak{R}(t)$ *and* $\Psi(t)$ *are independent for every* t.

Di erentiating to obtain the pdf's we get

$$p_{\mathfrak{R}}(r) = \frac{d}{dr}F_{\mathfrak{R}}(r) = \begin{cases} \frac{r}{\sigma_N^2} e^{-r^2/2\sigma_N^2}, & \text{if } r \ge 0, \\ 0, & \text{if } r < 0, \end{cases}$$

$$p_\Psi(\psi) = \frac{d}{d\psi}F_\Psi(\psi) = \begin{cases} \frac{1}{2\pi}, & \text{if } 0 \le \psi < 2\pi\ , \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof. ∎

Observe that the joint pdf is given by

$$p_{\mathfrak{R},\Psi}(r,\psi) = p_{\mathfrak{R}}(r)p_\Psi(\psi) = \begin{cases} \frac{r}{2\pi\sigma_N^2} e^{-r^2/2\sigma_N^2}, & \text{if } r \ge 0 \text{ and } 0 \le \psi < 2\pi\ , \\ 0, & \text{otherwise,} \end{cases}$$

by independence of $\mathfrak{R}(t)$ *and* $\Psi(t)$.